

# Numerically stable computation of circular visibility

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## Abstract

We address the question if a point inside a domain bounded by a simple closed arc spline is circularly visible from a specified arc from the boundary. We provide a simple and numerically stable linear time algorithm that solves this problem. In particular, we present an easy to check criterion that implies that a point is not visible from a specified boundary arc.

*Keywords:* Circular visibility, arc spline, channel

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## 1. Introduction

A point in the plane is called *circularly visible* inside a planar domain from another point if the two points can be connected by a circular arc that lies inside this domain. Algorithms that compute the set of all circularly visible points inside a polygon from a point or edge are well studied and there exist algorithms with a linear runtime, cf. [1, 2, 3]. The problem we want to consider here differs in two ways: we want to look at domains bounded by an *arc spline*, a curve that consists of circular arcs and line segments, and we only want to know if a point is visible from a specified arc on the boundary. We present a simple and numerically stable algorithm that decides in linear time if a point is circularly visible from a boundary arc inside a simple closed arc spline. For this purpose, we supply an easy to check criterion that directly implies that a point is not circularly visible from an arc.

We use this algorithm to improve the numerical stability of the SMAP approach, which computes an approximating smooth arc spline with the minimal number of segments within a specified maximal tolerance, cf. [6]. There, the basic task is closely related to the computation of the circular visibility set from a starting arc. It is known that the boundary of the circular visibility set consists of “boundary arcs” having three points in common with the boundary of the domain. Due to numerical inaccuracies, however, such boundary arcs might be missed. With the approach presented in this paper, we can determine if a point

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is visible and so we can localize the position of boundary arcs.

This paper is organized as follows: In Section 2, we introduce basic notations and definitions. In Section 3, we define a key tool for later proofs, a total order on a specified set of arcs. In Section 4, a sufficient condition for a point to be not circularly visible from an arc is shown. In Section 5, we present a linear time algorithm to decide if a point is circularly visible from an arc.

## 2. Notation and basic definitions

We call a continuous mapping  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  a *path* and  $\alpha(0)$  its *starting point* and  $\alpha(1)$  its *endpoint*. A path  $\alpha$  is *closed* if  $\alpha(0) = \alpha(1)$ , it is *simple* if it is injective and *simple closed* if it is closed and  $\alpha|_{[0,1]}$  is injective. Note that the image of a simple closed path is a *Jordan curve* which divides its complement, according to the Jordan curve theorem, into two connected components: a bounded one which we call the *interior* of the Jordan curve and an unbounded one, its *exterior*. As usual in the literature, we will use  $\alpha$  for both the mapping and the image, usually referred to as a *curve*. In particular, this allows us to write  $p \in \alpha$  instead of  $p \in \alpha([0, 1])$ .

We denote  $\alpha((0, 1))$  by  $\alpha^\circ$  and by  $\bar{\alpha}$  the *reverse path* defined by  $\bar{\alpha}(t) = \alpha(1 - t)$ . Let  $\alpha$  be a simple path and  $p \in \alpha$ . We denote by  $t_\alpha(p)$  the unique parameter in  $[0, 1]$  with  $\alpha(t_\alpha(p)) = p$ . We write  $t(p)$  if the corresponding path is clear from the context. For  $p, q \in \alpha$  we write  $p \prec_\alpha q$  if  $t_\alpha(p) < t_\alpha(q)$ .

A path  $\gamma$  of the form

$$\gamma(t) = c + r \cdot \begin{pmatrix} \cos(2\pi at + t_1) \\ \sin(2\pi at + t_1) \end{pmatrix}, \quad c \in \mathbb{R}^2, r > 0, a \in (0, 1), t_1 \in [0, 2\pi),$$

is called a *positively oriented arc*. We call the reverse path  $\bar{\gamma}$  of a positively oriented arc a *negatively oriented arc*. The path  $\ell$  defined by  $\ell(t) = (1 - t) \cdot p_1 + t \cdot p_2$ ,  $p_1, p_2 \in \mathbb{R}^2$ ,  $p_1 \neq p_2$ , is a *line segment from  $p_1$  to  $p_2$*  denoted by  $[p_1, p_2]$ . We call a path an *arc* if it is an arc of either orientation or a line segment. The set of all arcs will be denoted by  $\Gamma$ .

As an arc  $\gamma$  is differentiable with respect to  $t$  and its derivative  $\dot{\gamma}(t)$  does not vanish for any  $t \in [0, 1]$ , we can define the *unit tangent vector*  $\gamma' : [0, 1] \rightarrow S^1$ , where  $S^1$  is the unit sphere, by  $\gamma'(t) := \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|_2}$  and the *normal of length one "to the left"*  $\gamma^\perp(t) := (-v, u)^T$  with  $(u, v)^T = \gamma'(t)$ .

For  $p, q, r \in \mathbb{R}^2, \tau \in S^1$ , we denote by  $\gamma[p, r, q]$  the arc with starting point  $p$ , endpoint  $q$  that passes through  $r$  and by  $\gamma[\tau, p, q]$  the arc with starting point  $p$ , endpoint  $q$  and  $\tau$  as starting point tangent and by  $\gamma[p, q, \tau]$  the arc with starting point  $p$ , endpoint  $q$  and  $\tau$  as endpoint tangent. Note that  $\gamma[p, r, q]$  exists and is unique if  $p, q, r$  are distinct,  $q \notin [p, r]$  and  $p \notin [r, q]$ . Likewise,  $\gamma[\tau, p, q]$  and  $\gamma[p, q, \tau]$  exist and are unique if  $p \neq q$  and  $\tau$  and  $(p - q)$  are not pointing into the same direction.

Let  $\gamma$  be a positively or negatively oriented arc or a line segment, then we call  $[\gamma] := \gamma(\mathbb{R})$  the *corresponding circle* or the *corresponding line*, respectively.

Let  $\gamma_1, \gamma_2, \dots, \gamma_n$  be arcs with  $\gamma_k(1) = \gamma_{k+1}(0)$ ,  $k \in \{1, \dots, n-1\}$ . Then, we call the path  $\gamma_1 \sqcup \gamma_2 \sqcup \dots \sqcup \gamma_n$  defined as the concatenation

$$(\gamma_1 \sqcup \gamma_2 \sqcup \dots \sqcup \gamma_n)(t) := \gamma_k(nt - k + 1), \quad t \in \frac{1}{n}[k-1, k], \quad k = 1, 2, \dots, n$$

an *arc spline with  $n$  segments*. We call an arc spline *simple*, *closed* or *simple closed* if the corresponding path is simple, closed or simple closed, respectively. The points  $\gamma_1(0), \gamma_2(0), \dots, \gamma_n(0), \gamma_n(1)$  are called the *breakpoints* of the arc spline.

Let  $\ell$  be a line segment and  $p \in \mathbb{R}^2$ . A point  $p$  is *strictly left of*  $\ell$  if  $\langle \ell^\perp(0), p - \ell(0) \rangle > 0$  and it is *strictly right of*  $\ell$  if the inner product is negative. We say that  $p$  is *strictly left of* a positively oriented arc  $\gamma$  if  $p$  is in the interior of  $[\gamma]$ , it is *strictly left of* a negatively oriented arc  $\gamma$  if  $p$  is in the exterior of  $[\gamma]$ . Furthermore,  $p$  is left of an arc  $\gamma$  if it is either strictly left of  $\gamma$  or  $p \in [\gamma]$ . With  $p \in \gamma^\circ$ , a set  $M \subset \mathbb{R}^2$  is said to be *locally left of  $\gamma$  at  $p$*  if there is an  $\varepsilon > 0$  so that for every  $\delta \in (0, \varepsilon)$  the set  $M \cap B_p(\delta)$ , with  $B_p(\delta) := \{x \in \mathbb{R}^2 : \|x - p\|_2 < \delta\}$ , is nonempty and every  $q \in M \cap B_p(\delta)$  is left of  $\gamma$ . We say that  $M$  is *locally left of  $\gamma$*  if for every  $p \in \gamma^\circ$  it is locally left of  $\gamma$  at  $p$ .

Let  $\gamma$  be an arc,  $\alpha$  a path and  $t \in [0, 1]$  with  $\alpha(t) \in \gamma^\circ$ . We say  $\alpha$  *leaves  $\gamma$  in  $t$  to the left* if  $\alpha(t + \varepsilon)$  is strictly left of  $\gamma$  for every sufficiently small  $\varepsilon > 0$ . Likewise, we say  $\alpha$  *approaches  $\gamma$  in  $t$  from the left* if  $\alpha(t - \varepsilon)$  is strictly left of  $\gamma$  for every sufficiently small  $\varepsilon > 0$ . Likewise, the definitions hold for “right” instead of “left”.

We say that  $\alpha$  *cuts  $\gamma$  in  $t$  from the left* if there is a  $t' \in [0, t]$  with  $\alpha([t', t]) \in \gamma$  such that  $\alpha$  approaches  $\gamma$  in  $t'$  from the left and it leaves  $\gamma$  in  $t$  to the right. Likewise, we define a *cut from the right*. Note that if  $\alpha$  is an arc then  $\alpha$  cuts  $\gamma$  in  $t$  from the right if and only if  $\langle \gamma^\perp(t_\gamma(\alpha(t))), \alpha'(t) \rangle > 0$ . Hence, this definition is consistent with the usual intuition of cutting from the left or right. Furthermore, this yields that an arc  $\gamma_1$  cuts another arc  $\gamma_2$  from the left if and only if  $\gamma_2$  cuts  $\gamma_1$  from the right. We use

$$\begin{aligned} \alpha \cap_- \gamma &:= \{t \in [0, 1] : \alpha \text{ cuts } \gamma \text{ in } t \text{ from the left} \}, \\ \alpha \cap_+ \gamma &:= \{t \in [0, 1] : \alpha \text{ cuts } \gamma \text{ in } t \text{ from the right} \} \end{aligned}$$

**Definition 1 (Channel).** Let  $\sigma$  be an arc and  $\kappa = \kappa_1 \sqcup \kappa_2 \sqcup \dots \sqcup \kappa_n$  an arc spline with arcs  $\kappa_1, \dots, \kappa_n$  such that  $\sigma \sqcup \kappa$  is simple closed. Furthermore, we demand that the closure of the interior of  $\sigma \sqcup \kappa$ , denoted by  $P$ , is locally left of  $\sigma$  and that  $\langle \sigma^\perp(1), \kappa'(0) \rangle > 0$  and  $\langle \sigma^\perp(0), \kappa'(1) \rangle < 0$ . We call  $\sigma$  the *starting arc* of the *channel*  $P$ ,  $\kappa$  is called the *channel boundary* and  $\kappa_j, j \in \{1, \dots, n\}$  are called *channel segments*.

Channels are a standard tool to approximate data within a certain maximal tolerance. It is usual to use polygonal channels and there are efficient methods to construct them, cf. [5]. Here, however, we use arc splines as channel boundary as this is necessary in the SMAP approach, cf. [6]. The constraint for the unit

tangent vector of  $\kappa$  in 0 and 1 is actually not necessary but it simplifies some of the following concepts.

**Definition 2 (Circular Visibility).** We say that a point  $p \in P^\circ$  is (*circularly*) *visible* if there is an arc  $\gamma$  with starting point on  $\sigma$ , endpoint  $p$  and  $\gamma \subset P$ . In this case, we call  $\gamma$  a *visibility arc*.

The goal of the paper is an algorithm that either computes a visibility arc or proves that the point is not visible. An example showing a channel and a visibility arc can be seen in Figure 2 on page 9.

### 3. Connecting arcs and total order

We know that a point  $p$  is circularly visible from the starting arc  $\sigma$  if there exists a visibility arc, that is, an arc inside the channel with starting point on  $\sigma$  and endpoint  $p$ . In this chapter, we will study the set of candidates for visibility arcs, the so-called connecting arcs. These are arcs with starting point on  $\sigma$  and endpoint  $p$ , but we do not yet care if they are inside the channel. We will define a total order on this set of arcs which will provide a helpful tool to find a visibility arc and to prove a criterion that classifies a point as not visible. In this chapter we assume that  $\sigma$  is an arc and  $p \in \mathbb{R}^2 \setminus \sigma$ .

**Definition 3 (Connecting Arc).** Let  $\sigma$  be an arc and  $p \in \mathbb{R}^2 \setminus \sigma$ . If  $p \notin [\sigma]$  then let

$$\Gamma^*(\sigma, p) := \{ \gamma \in \Gamma : \gamma(0) \in \sigma^\circ, \gamma \cap \sigma = \{\gamma(0)\}, \gamma(1) = p \\ \text{and } \gamma \text{ leaves } \sigma \text{ in } 0 \text{ to the left} \}$$

and  $\Gamma(\sigma, p) := \overline{\Gamma^*(\sigma, p)}$ . Otherwise, let

$$\Gamma(\sigma, p) := \{ \gamma[\tau, \sigma(t), p] : t \in [0, 1], \tau \in S^1, \langle \sigma^\perp(t), \tau \rangle > 0 \} \\ \cup \{ \gamma[-\sigma'(0), \sigma(0), p], \gamma[\sigma'(1), \sigma(1), p] \}$$

We call an arc  $\gamma \in \Gamma(\sigma, p)$  a *connecting arc (from  $\sigma$  to  $p$ )*.

Let  $q, r \in \mathbb{R}^2 \setminus \sigma$  and  $q, r$  and  $p$  pairwise distinct. If there exists a unique  $\gamma \in \Gamma(\sigma, p)$  with  $q, r \in \gamma$ ,  $q \prec_\gamma r$ , then we denote  $\gamma$  by  $\gamma_\sigma[q, r, p]$ .

Note that for any  $q, r \in \mathbb{R}^2 \setminus \sigma$  with  $q, r$  and  $p$  pairwise distinct the arc  $\gamma_\sigma[q, r, p]$  is unique if it exists, as  $q, r, p$  defines a unique circle and there is at most one starting point on  $\sigma$  such that the arc starts to the left.

**Remark 4.** For continuity reasons, we use in Definition 5 the following extension of directional cuts. Note that with the definition from above one arc can not cut another one at its starting point or endpoint. Let  $\gamma_1, \gamma_2 \in \Gamma(\sigma, p)$ , especially we have  $\gamma_1(0), \gamma_2(0) \in \sigma$ , and let  $\gamma_1(0) \prec_\sigma \gamma_2(0)$ . We say that  $\gamma_1$  cuts  $\gamma_2$  in 1 from the left and  $\gamma_2$  cuts  $\gamma_1$  in 1 from the right if  $\gamma_1'(1) = \gamma_2'(1)$  and every  $\gamma_3 \in \Gamma(\sigma, p)$  with  $\gamma_1(0) \prec_\sigma \gamma_3(0) \prec_\sigma \gamma_2(0)$  and  $\gamma_1 \cap \gamma_3 = \gamma_2 \cap \gamma_3 = \{p\}$  satisfies  $\gamma_3'(1) = \gamma_1'(1)$ . If  $q := \gamma_1(0) \in \gamma_2$ , then we say  $\gamma_1 \cap_- \gamma_2 = \{0\}$  and

$\gamma_2 \cap_+ \gamma_1 = \{t_2(q)\}$ . Likewise, if  $q := \gamma_2(0) \in \gamma_1$  and  $\gamma_1(0) \notin \gamma_2$ , then we say  $\gamma_1 \cap_- \gamma_2 = \{t_1(q)\}$  and  $\gamma_2 \cap_+ \gamma_1 = \{0\}$ . The last two cases are only possible if  $p$  is strictly right of  $\sigma$ .

**Definition 5.** Let  $\sigma \in \Gamma$ ,  $p \in \mathbb{R}^2 \setminus \sigma$  and  $\gamma_1, \gamma_2 \in \Gamma(\sigma, p)$ . We say  $\gamma_1 \leq \gamma_2$  if one of the following conditions holds:

1.  $\gamma_1(0) \prec_\sigma \gamma_2(0)$  and  $\gamma_1 \cap_- \gamma_2 = \emptyset$ ,
2.  $\gamma_1(0) \succ_\sigma \gamma_2(0)$  and  $\gamma_2 \cap_- \gamma_1 \neq \emptyset$ ,
3.  $\gamma_1(0) = \gamma_2(0) =: \sigma(t^*)$  and  $\langle \sigma'(t^*), \gamma_1'(0) \rangle \leq \langle \sigma'(t^*), \gamma_2'(0) \rangle$ .

An illustration of the three different cases can be found in Figure 1.

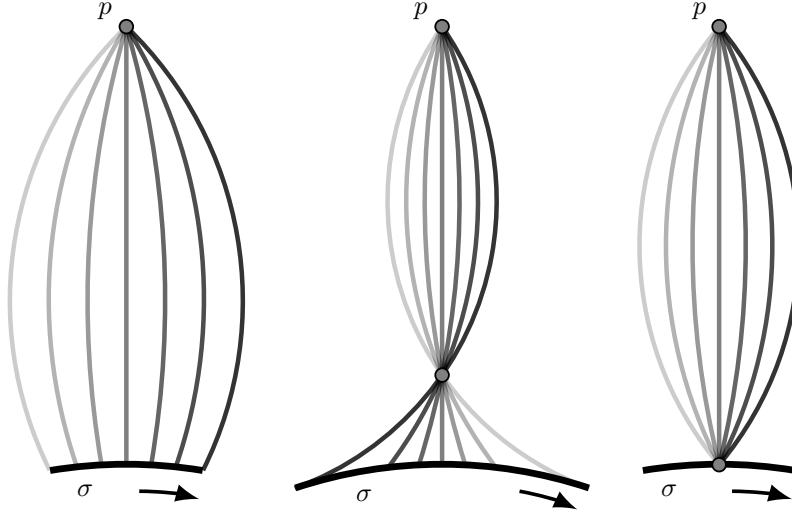


Figure 1: Illustration of the cases 1. (left), 2. (middle) and 3. (right) of Definition 5. An arc is printed the darker the greater it is.

**Theorem 6.** Let  $\sigma$  be an arc and  $p \in \mathbb{R}^2 \setminus \sigma$ . Then “ $\leq$ ” is a total order on  $\Gamma(\sigma, p)$ .

PROOF. Reflexivity, totality and antisymmetry are immediate. In order to prove the transitivity of “ $\leq$ ”, let  $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(\sigma, p)$  be pairwise distinct with  $\gamma_1 \leq \gamma_2$  and  $\gamma_2 \leq \gamma_3$ . We distinguish a total of six different cases based on the possible order of the starting points of  $\gamma_1, \gamma_2$  and  $\gamma_3$  on  $\sigma$ . We denote by  $t_1, t_2, t_3 \in [0, 1]$  the parameters such that  $\sigma(t_1) = \gamma_1(0), \sigma(t_2) = \gamma_2(0)$  and  $\sigma(t_3) = \gamma_3(0)$ .

First, let  $t_1 \leq t_2 \leq t_3$ . If  $t_1 = t_3$  then  $t_1 = t_2 = t_3$  and we get  $\langle \sigma'(t_1), \gamma_1'(0) \rangle \leq \langle \sigma'(t_1), \gamma_2'(0) \rangle \leq \langle \sigma'(t_1), \gamma_3'(0) \rangle$  which yields  $\gamma_1 \leq \gamma_3$ .

So, let  $t_1 \neq t_3$  and assume the contrary,  $\gamma_3 < \gamma_1$ . By definition,  $\gamma_1 \cap_- \gamma_3 \neq \emptyset$ , so

define  $\gamma_1 \cap_- \gamma_3 =: \{t_{13}\}$  and  $\gamma_3 \cap_+ \gamma_1 =: \{t_{31}\}$ . We consider the simple closed path

$$\alpha := \sigma|_{[t_1, t_3]} \sqcup \gamma_3|_{[0, t_{31}]} \sqcup \overline{\gamma_1|_{[0, t_{13}]}}$$

and let  $Z$  be the unique connected component in  $\mathbb{R}^2 \setminus \alpha$  that is locally left of  $\sigma|_{[t_1, t_3]}$ . Then,  $Z$  is locally right of  $\gamma_1|_{[0, t_{13}]}$  and locally left of  $\gamma_3|_{[0, t_{31}]}$  and  $\gamma_2$  starts into  $Z$  meaning that there is an  $\varepsilon > 0$  so that  $\gamma_2(t) \in Z$  for every  $t \in (0, \varepsilon)$ . As  $\gamma_3$  cuts  $\gamma_1$  in  $t_{31}$  from the right we know that if  $t_{31} \neq 1$  then the endpoint  $p$  of  $\gamma_3$  is not in the closure of  $Z$ . Hence,  $\gamma_2$  either has to cut  $\gamma_1$  from the right or  $\gamma_3$  from the left. If  $t_{31} = 1$ , with the extensions of cuts for connecting arcs, cf. Remark 4, we get analogously that  $\gamma_2$  cuts  $\gamma_1$  from the right or  $\gamma_3$  from the left. Either way, this yields  $\gamma_2 < \gamma_1$  or  $\gamma_3 < \gamma_2$  which is a contradiction.

We now turn to the case  $t_3 \leq t_2 \leq t_1$ ,  $t_1 \neq t_3$ . If  $t_1 = t_2$  we define  $t_{12} := t_{21} := 0$ , otherwise we know that  $\gamma_2 \cap_- \gamma_1 \neq \emptyset$  and we define  $\gamma_2 \cap_- \gamma_1 =: \{t_{21}\}$  and  $\gamma_1 \cap_+ \gamma_2 =: \{t_{12}\}$ . Likewise, we define  $t_{23} := t_{32} := 0$  if  $t_2 = t_3$ ,  $\gamma_3 \cap_- \gamma_2 =: \{t_{32}\}$  and  $\gamma_2 \cap_+ \gamma_3 =: \{t_{23}\}$ , otherwise. We assume  $t_{21} \leq t_{23}$ , otherwise the proof works analogously. Similarly to the first case, consider the simple closed path

$$\alpha := \sigma|_{[t_3, t_2]} \sqcup \gamma_2|_{[0, t_{23}]} \sqcup \overline{\gamma_3|_{[0, t_{32}]}}$$

and let  $Z$  be the connected component in  $\mathbb{R}^2 \setminus \alpha$  that is locally left of  $\sigma|_{[t_3, t_2]}$ . The case  $t_{21} = t_{23} = 1$  is easily verified. In the case  $t_{21} = t_{23} \neq 1$  we know that  $\gamma_1$  cuts  $\gamma_3$  in  $t_{12}$  since  $\gamma_1 \neq \gamma_3$ . If  $\gamma_1$  cuts  $\gamma_3$  in  $t_{12}$  from the left then  $\gamma_1$  starts in  $t_{12}$  into  $Z$ . This would yield at least three intersections of  $\gamma_1$  with either  $\gamma_2$  or  $\gamma_3$  which would be a contradiction since then the arcs would coincide. Hence,  $\gamma_1$  intersects  $\gamma_3$  in  $t_{12}$  from the right and we have  $\gamma_1 \leq \gamma_3$ . If  $t_{21} < t_{23}$  then  $\gamma_1$  starts in  $t_{12}$  into  $Z$  as  $Z$  is locally left of  $\gamma_2|_{[0, t_{23}]}$  and  $\gamma_1$  cuts  $\gamma_2|_{[0, t_{23}]}$  in  $t_{12}$  from the right. As in the first case, we can conclude that  $\gamma_1$  has to cut  $\gamma_3|_{[0, t_{32}]}$  from the right or  $\gamma_2|_{[0, t_{23}]}$  from the left. As  $\gamma_1$  and  $\gamma_2$  can not have three intersections,  $\gamma_1$  has to cut  $\gamma_3$  from the right and we have  $\gamma_1 \leq \gamma_3$ .

Now, let us consider the remaining four possibilities how to order  $t_1, t_2, t_3$ . We prove this cases by contradiction, so assume  $\gamma_3 < \gamma_1$ . In the case  $t_1 \leq t_3 \leq t_2$ , with  $\gamma_2 \leq \gamma_3$ , we are in the setting of the second case and this yields  $\gamma_2 < \gamma_1$ , a contradiction. If  $t_2 \leq t_1 \leq t_3$  then with  $\gamma_1 \leq \gamma_2$  we also are in the setting of the second case and get  $\gamma_3 < \gamma_2$ . If  $t_2 \leq t_3 \leq t_1$  or  $t_3 \leq t_1 \leq t_2$  then with  $\gamma_2 \leq \gamma_3$  or  $\gamma_1 \leq \gamma_2$  we are in the setting of the first case and get  $\gamma_2 < \gamma_1$  or  $\gamma_3 < \gamma_2$ , respectively. Hence, in either case we get a contradiction.  $\square$

For the rest of the chapter we treat the problem of computing a maximal or minimal connecting arc with respect to “ $\leq$ ” from Definition 5. We use this in the initialization of Algorithm 1 in Section 5. Assume that  $p_1, p_2, p_3$  are three pairwise distinct points. We know that there exists a unique arc with starting point  $p_1$ , endpoint  $p_3$  that passes  $p_2$  unless  $p_1 \in [p_2, p_3]$  or  $p_3 \in [p_1, p_2]$ . Because of Remark 9 we can ignore these cases and we will assume that every such arc exists. Likewise, we assume that for any two points  $p_1, p_2$ ,  $p_1 \neq p_2$  and any

direction  $\tau \in S^1$  the arc with starting point  $p_1$ ,  $\tau$  as unit tangent vector at the start and endpoint  $p_2$  exists.

**Proposition 7.** *Let  $\sigma$  be an arc and  $p \in \mathbb{R}^2 \setminus \sigma$ . If  $p$  is strictly left of  $\sigma$  or  $p \in [\sigma]$  then  $\gamma[\sigma'(1), \sigma(1), p]$  is the maximal connecting arc in  $\Gamma(\sigma, p)$  with respect to “ $\leq$ ”. If  $p$  is strictly right of  $\sigma$  then the maximal connecting arc in  $\Gamma(\sigma, p)$  with respect to “ $\leq$ ” is  $\gamma[\sigma(0), \sigma(1), p]$ .*

PROOF. First, let  $p$  be strictly left of  $\sigma$  and let  $\gamma_1 = \gamma[\sigma'(1), \sigma(1), p]$ . Then,  $\gamma_1$  is in the boundary of  $\Gamma^*(\sigma, p)$  and by Definition 3,  $\gamma_1$  is a connecting arc. To show that  $\gamma_1$  is maximal we take any  $\gamma_2 \in \Gamma(\sigma, p)$  and show  $\gamma_2 \leq \gamma_1$ . If  $\gamma_2(0) = \sigma(1)$  then  $\langle \sigma'(1), \gamma_2'(0) \rangle \leq 1 = \langle \sigma'(1), \gamma_1'(0) \rangle$  which proves  $\gamma_2 \leq \gamma_1$ . Now, let  $\gamma_2(0) \neq \sigma(1)$  and assume the contrary,  $\gamma_2 > \gamma_1$ . As  $\gamma_2(0) \prec_\sigma \sigma(1) = \gamma_1(0)$ ,  $\gamma_2 > \gamma_1$  yields that  $\gamma_2$  cuts  $\gamma_1$  from the left. As  $p$  is strictly left of  $\sigma$  and  $\gamma_1'(0) = \sigma'(1)$ , we know that  $\sigma$  is right of  $\gamma_1$ . Hence,  $\gamma_2$  starts strictly right of  $\gamma_1$  which yields that  $\gamma_2$  must cut  $[\gamma_1]$  from the right before  $\gamma_2$  can cut  $\gamma_1$  from the left. This is a contradiction to  $\gamma_1 \neq \gamma_2$  as it would yield at least three intersections.

Now, let  $p$  be strictly right of  $\sigma$  and let  $\gamma_1 = \gamma[\sigma(0), \sigma(1), p]$ . Again,  $\gamma_1$  is a connecting arc as it is in the boundary of  $\Gamma^*(\sigma, p)$ . Denote by  $\alpha$  the simple closed path  $\gamma_1|_{[0, t_1(\sigma(1))]} \sqcup \bar{\sigma}$  and let  $Z$  be the connected component of  $\mathbb{R}^2 \setminus \alpha$  that is locally left of  $\sigma$ . Let  $\gamma_2 \in \Gamma(\sigma, p)$ . If  $\gamma_2(0) = \sigma(0)$ , then we know that it does not start into  $Z$  as otherwise  $\gamma_2$  would have to cut  $\sigma^\circ$  or  $\gamma_1$  three times, as  $p \notin Z$ . Hence, if  $\gamma_2(0) = \sigma(0)$  then  $\langle \sigma'(0), \gamma_2'(0) \rangle \leq \langle \sigma'(0), \gamma_1'(0) \rangle$ . If  $\gamma_2(0) \in \sigma^\circ$  then  $\gamma_2$  starts into  $Z$  and as  $p \notin Z$  and  $Z$  is locally right of  $\gamma_1|_{[0, t_1(\sigma(1))]}$  we know that  $\gamma_2$  cuts  $\gamma_1$  from the right, which yields  $\gamma_2 \leq \gamma_1$ . If  $\gamma_2(0) = \sigma(1)$ , then by Remark 4  $\gamma_2$  cuts  $\gamma_1$  in 0 from the right and therefore  $\gamma_2 \leq \gamma_1$ .

Now, let  $p \in [\sigma]$ ,  $\gamma_1 = \gamma[\sigma'(1), \sigma(1), p]$  and  $\gamma_2 \in \Gamma(\sigma, p)$ . If  $\gamma_2(0) = \sigma(1)$  then  $\langle \sigma'(1), \gamma_2'(0) \rangle \leq 1 = \langle \sigma'(1), \gamma_1'(0) \rangle$ , so we get  $\gamma_2 \leq \gamma_1$ . If  $\gamma_2 = \gamma[-\sigma'(0), \sigma(0), p]$  then  $\gamma_1 \cap \gamma_2 = \{p\}$  and  $\gamma_1'(1) = -\gamma_2'(1)$ , which yields  $\gamma_2 \cap_- \gamma_1 = \emptyset$ , so  $\gamma_2 < \gamma_1$ . Otherwise, we know that  $\langle \sigma^\perp(t_\sigma(\gamma_2(0))), \gamma_2'(0) \rangle > 0$ , so  $[\gamma_2] \neq [\sigma] = [\gamma_1]$ . As  $\gamma_2(0), \gamma_2(1) \in [\gamma_1]$ , there can not be a third point that is both on  $\gamma_2$  and on  $[\gamma_1]$  and we know that  $\gamma_1'(1) \neq \gamma_2'(1)$ . This yields  $\gamma_2 \cap_- \gamma_1 = \emptyset$  and we get  $\gamma_2 < \gamma_1$ .  $\square$

**Remark 8.** Similarly, one can show that an arc  $\gamma_1$  with  $\gamma_1'(0) = \sigma'(t_\sigma(\gamma_1(0)))$  is greater than any arc  $\gamma_2$  with  $\gamma_2(0) \preceq_\sigma \gamma_1(0)$ .

**Remark 9.** If  $p$  is left of  $\sigma$ , then a maximal connecting arc obviously exists unless  $\sigma'(1)$  and the normalized vector  $\sigma(1) - p$  are equal. If  $p$  is strictly right of  $\sigma$ , then a maximal connecting arc exists unless  $p$  is on the line segment  $[\sigma(0), \sigma(1)]$  or  $\sigma(0)$  is on the line segment  $[p, \sigma(1)]$ . We know that the channel has finite diameter and so the length of any arc  $\gamma \subset P$  is bounded from above. This allows us to ignore arcs exceeding a certain maximal length and to replace them by a respective arc having this maximal length. In case of a maximal connecting arc, the maximal connecting arc with restricted length will still be large enough

for all requirements. Hence, we will simply assume that the maximal connecting arc always exists.

#### 4. Restrictions

In this chapter,  $\sigma$  will denote the starting arc and  $\kappa$  the boundary of the channel  $P$  as stated in Definition 1 and we have  $p \in P^\circ$ .

Restrictions will be the key tool to characterize circular visibility. Most papers on circular visibility use the well-known fact that the boundary of the set of circularly visible points from a given point or edge consists of arcs having two, or, in the case of the edge, three points in common with the channel boundary, cf. [1, 2, 3, 6]. An example is shown in Figure 2, in which the boundary arc of the visibility set is shown dashed. We will use a similar approach, but we extend the definition of restrictions to arcs that are not completely inside the channel. An example can be seen in Figure 3: we have a connecting arc  $\gamma$  with endpoint  $p \in P^\circ$  but it is not a visibility arc as  $\gamma \not\subset P$ . The question we are interested in is if we can modify  $\gamma$  such that we obtain a visibility arc ending in  $p$  or not.

The point  $r_1$  shows an example of a restriction from the left in the ‘classical’ sense:  $\gamma$  would immediately leave the channel if we would move it slightly to the left at  $r_1$ .

The situation at the point  $r_2$  is slightly different:  $\gamma$  is leaving the channel in this region. Nevertheless, since  $\gamma$  would run within the channel if we would push it to the left there, we will call this also a *restriction*; certainly ‘from the right’ in this case, however. The part of the channel, where the arc is running outside the channel, shown as thick line, will be called a *violation*.

**Remark 10.** For continuity reasons, we extend the definition of directional cuts for the channel boundary. Let  $\gamma \in \Gamma(\sigma, p)$ . We say that  $\kappa$  approaches  $\gamma$  in 0 from the right if  $\kappa(0) \in \gamma$ . If  $\gamma(0) = \kappa(0)$  then we say that  $\kappa$  leaves  $\gamma$  in 0 to the left or right if it leaves  $[\gamma]$  in 0 to the left or right, respectively. Similarly, if  $\gamma(1) = \kappa(1)$  then we say that  $\kappa$  approaches  $\gamma$  in 1 from the left or right if it approaches  $[\gamma]$  in 1 from the left or right, respectively.

**Definition 11 (Restriction).** Let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be an arc spline and  $\gamma$  an arc. Then we define  $\Delta(\gamma, \alpha) \in \mathbb{Z}$  by

$$\begin{aligned} \Delta(\gamma, \alpha) := & |\{t \in [0, 1] : \alpha \text{ approaches } \gamma \text{ in } t \text{ from the right}\}| \\ & - |\{t \in [0, 1] : \alpha \text{ approaches } \gamma \text{ in } t \text{ from the left}\}| \\ & + |\{t \in [0, 1] : \alpha \text{ leaves } \gamma \text{ in } t \text{ to the left}\}| \\ & - |\{t \in [0, 1] : \alpha \text{ leaves } \gamma \text{ in } t \text{ to the right}\}|. \end{aligned}$$

With  $p \in P^\circ$ ,  $\gamma \in \Gamma(\sigma, p)$  and  $q \in \kappa$ , we define

$$\Delta_q(\gamma, \kappa) := \Delta(\gamma, \kappa|_{[0, t_\kappa(q)]}).$$



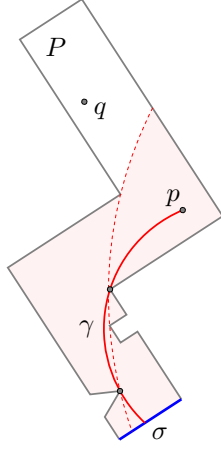


Figure 2: Channel  $P$  with starting arc  $\sigma$ ; set of circularly visible points shaded; visible point  $p$  with visibility arc  $\gamma$ ;  $q$  is not visible

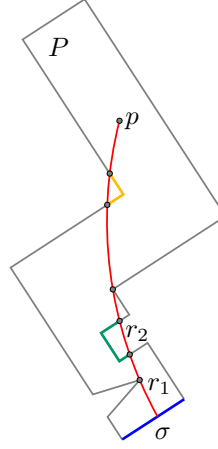


Figure 3: Example showing restrictions and violations of a connecting arc

We call  $q \in \kappa$  a *restriction from the right of  $\gamma$*  if  $\Delta_q(\gamma, \kappa) = 1$ , a *restriction from the left of  $\gamma$*  if  $\Delta_q(\gamma, \kappa) = -1$ , a *violation from the right of  $\gamma$*  if  $\Delta_q(\gamma, \kappa) > 1$  and a *violation from the left of  $\gamma$*  if  $\Delta_q(\gamma, \kappa) < -1$ .

Furthermore, we call  $q = \gamma(0)$  a *starting restriction from the left* if  $\sigma$  is locally right of  $[\gamma]$  at  $\gamma(0)$  and we call it a *starting restriction from the right* if  $\sigma$  is locally left of  $[\gamma]$  at  $\gamma(0)$ .

One advantage of this approach with respect to numerical stability can be seen directly from the definition. Restrictions in the ‘usual meaning’ are points where the channel touches the arc but the arc is a subset of the channel. Hence, we have a equality constraint. In definition 11 equality is not necessary. If the arc slightly leaves the channel in a certain area, we have a restriction and a violation around there. So, we can either try to shrink the violation by updating the arc appropriately or if the violation is smaller than a certain tolerance then we can simply ignore it.

If  $\alpha$  is a simple closed arc spline and  $\gamma$  is an arc with  $\gamma(0), \gamma(1) \notin \alpha$  then the following lemma shows that the value of  $\Delta(\gamma, \alpha)$  can be determined just by analyzing in which connected component of  $\mathbb{R}^2 \setminus \alpha$  the points  $\gamma(0)$  and  $\gamma(1)$  are. Note that this result is connected to the behavior of the winding number along a path, known from algebraic topology or complex analysis, see [7].

**Proposition 12.** *Let  $\alpha$  be a simple closed arc spline whose interior  $I_\alpha$  is locally left of  $\alpha$  and  $\gamma$  an arc with  $\gamma(0), \gamma(1) \notin \alpha$ .*

*If  $\gamma(0), \gamma(1) \in I_\alpha$  or  $\gamma(0), \gamma(1) \notin I_\alpha$  then  $\Delta(\gamma, \alpha) = 0$ . If  $\gamma(0) \in I_\alpha$  and  $\gamma(1) \notin I_\alpha$  then  $\Delta(\gamma, \alpha) = 2$  and if  $\gamma(0) \notin I_\alpha$  and  $\gamma(1) \in I_\alpha$  then  $\Delta(\gamma, \alpha) = -2$ .*

PROOF. Let  $\beta_1, \beta_2, \dots, \beta_n$  be arcs such that  $\beta := \gamma \sqcup \beta_1 \sqcup \dots \sqcup \beta_n$  is simple closed

and for  $i \in \{1, \dots, n\}$  we have  $\beta_i \cap \alpha = \{p_i\}$ ,  $p_i \in \beta_i^\circ$ ,  $p_i$  is not a breakpoint of  $\alpha$  and  $\alpha$  cuts  $\beta_i$  in  $t_\alpha(p_i)$  either from the left or from the right. To simplify notation we write  $\beta_0 := \gamma$ .

First, we show  $\sum_{i=0}^n \Delta(\beta_i, \alpha) = 0$ . We can always assume that  $\alpha(0) \notin \beta$  as  $\alpha$  is closed and we define  $0 = t_1 < t_2 < \dots < t_m = 1$  such that for every  $j \in \{1, \dots, m-1\}$  we have  $\alpha(t_j) \notin \beta$  and  $\{t \in [t_j, t_{j+1}] : \alpha(t) \in \beta\}$  is a nonempty interval. As  $\beta$  is simple closed, the interior  $I_\beta$  and the exterior  $E_\beta$  of  $\beta$  are well-defined and we know that either  $I_\beta$  is locally left of  $\beta_j$  and  $E_\beta$  is locally right of  $\beta_j$  for every  $j \in \{0, \dots, n\}$  or vice versa. We assume that  $I_\beta$  is locally left of  $\beta_0$ , since the arguments work analogously, otherwise. Let  $j \in \{1, \dots, m-1\}$ . If  $\alpha(t_j) \in I_\beta$  and  $\alpha(t_{j+1}) \in E_\beta$  then we know that there is exactly one  $t \in (t_j, t_{j+1})$ , one  $t' \in (t_j, t_{j+1})$  and one  $i \in \{0, \dots, n\}$  such that  $\alpha$  approaches  $\beta_i$  in  $t$  from the left and  $\alpha$  leaves  $\beta_i$  in  $t'$  to the right. With this unique index  $i$ , we know that  $\Delta(\beta_k, \alpha|_{[t_j, t_{j+1}]}) = -2$  if  $k = i$  and that it vanishes, otherwise. This yields  $\sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]}) = -2$ . Analogously, we get  $\sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]}) = 2$  if  $\alpha(t_j) \in E_\beta$  and  $\alpha(t_{j+1}) \in I_\beta$  and we get that  $\sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]})$  vanishes if both  $\alpha(t_j)$  and  $\alpha(t_{j+1})$  are either in  $I_\beta$  or in  $E_\beta$ . As  $\alpha$  is closed and  $\alpha(t_m) = \alpha(t_1)$  there are just as many  $j \in \{1, \dots, m-1\}$  with  $\sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]}) = -2$  as with  $\sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]}) = 2$ . This yields  $\sum_{i=0}^n \Delta(\beta_i, \alpha) = \sum_{j=1}^{m-1} \sum_{i=0}^n \Delta(\beta_i, \alpha|_{[t_j, t_{j+1}]}) = 0$ .

Now, let  $i \in \{1, \dots, n\}$ ,  $\beta_i(0) \in I_\alpha$  and let  $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \dots \sqcup \alpha_m$  with arcs  $\alpha_1, \dots, \alpha_m, m \in \mathbb{N}$ . We denote the exterior of  $\alpha$  by  $E_\alpha$ . Since  $\alpha$  cuts  $\beta_i$  exactly once from the left or from the right and since no intersection is a breakpoint of  $\beta$  or  $\alpha$ , we know that  $\beta_i(1) \in E_\alpha$  and that there is a unique  $j \in \{1, \dots, m\}$  such that  $\beta_i \cap \alpha_j \neq \emptyset$ . As  $I_\alpha$  is locally left of  $\alpha_j$ , we know that  $\beta_i$  cuts  $\alpha_j$  from the left. Hence, there is exactly one  $t \in [0, 1]$  such that  $\alpha_j(t) \in \beta_i$  and  $\alpha_j$  cuts  $\beta_i$  in  $t$  from the right. This yields  $\Delta(\beta_i, \alpha) = \Delta(\beta_i, \alpha_j) = 2$ . Analogously, we get  $\Delta(\beta_i, \alpha) = -2$  if  $\beta_i(0) \in E_\alpha$ .

Assume  $\gamma(0) \in I_\alpha$  and  $\gamma(1) \in E_\alpha$ . Then,  $\beta_1(0) \in E_\alpha, \beta_n(1) \in I_\alpha$  and  $\beta_{i+1}(0) = \beta_i(1) \in I_\alpha$  if  $\beta_i(0) \in E_\alpha, i \in \{1, \dots, n-1\}$  and vice versa. Hence,  $\Delta(\gamma, \alpha) = \Delta(\beta_0, \alpha) = -\sum_{i=1}^n \Delta(\beta_i, \alpha) = -(-2 + 2 - \dots + 2 - 2) = 2$ . Similarly, one can prove the remaining three cases.  $\square$

**Lemma 13.** *Let  $p \in P^\circ$  and  $\gamma \in \Gamma(\sigma, p)$ . We have  $\Delta(\gamma, \kappa) = 0$  if  $\kappa(1) \notin \gamma$  and  $\Delta(\gamma, \kappa) = -1$ , otherwise.*

PROOF. First, assume  $\gamma(0) \in \sigma^\circ$ . Then there exists an  $\varepsilon > 0$  such that  $\gamma([0, \varepsilon]) \cap \kappa = \emptyset$  and we know that  $\gamma(\varepsilon)$  and  $\gamma(1)$  are in  $P^\circ$ , that is in the same connected component of  $\mathbb{R}^2 \setminus (\sigma \sqcup \kappa)$ . With Proposition 12 we get  $\Delta(\gamma|_{[\varepsilon, 1]}, \sigma \sqcup \kappa) = 0$ . If  $\gamma|_{[\varepsilon, 1]} \cap \sigma = \emptyset$  then we get  $\Delta(\gamma|_{[\varepsilon, 1]}, \kappa) = 0$  and finally  $\Delta(\gamma, \kappa) = 0$ . Otherwise, we know that  $\sigma$  approaches  $\gamma|_{[\varepsilon, 1]}$  in 1 from the right if  $\sigma(1) \in \gamma|_{[\varepsilon, 1]}$  and that it leaves  $\gamma|_{[\varepsilon, 1]}$  in 0 to the left if  $\sigma(0) \in \gamma|_{[\varepsilon, 1]}$ . This fits the special cases in Remark 10 and we get  $\Delta(\gamma, \kappa) = 0$  if  $\kappa(1) \notin \gamma$  and  $\Delta(\gamma, \kappa) = -1$ , otherwise. Now, assume  $\gamma(0) = \sigma(1)$ . If  $\kappa$  leaves  $[\gamma]$  in 0 to the right then there exists an

$\varepsilon > 0$  such that  $\gamma((0, \varepsilon)) \cap \kappa = \emptyset$  and  $\gamma(\varepsilon) \in P^\circ$ . As in the first case, we get  $\Delta(\gamma|_{[\varepsilon, 1]}, \kappa) = 0$ . With Remark 10 we get  $\Delta(\gamma|_{[0, \varepsilon]}, \kappa) = 0$ , thus  $\Delta(\gamma, \kappa) = 0$ . If  $\kappa$  leaves  $[\gamma]$  in 0 to the left then there exists an  $\varepsilon > 0$  such that  $\gamma((0, \varepsilon)) \cap \kappa = \emptyset$  and  $\gamma(\varepsilon) \in \mathbb{R}^2 \setminus P$ . As Proposition 12 yields  $\Delta(\gamma|_{[\varepsilon, 1]}, \sigma \sqcup \kappa) = -2$ , we get  $\Delta(\gamma|_{[\varepsilon, 1]}, \kappa) = -2$ , analogously to the first case. With Remark 10 we get  $\Delta(\gamma, \kappa) = 0$ . If  $\kappa$  neither leaves  $\gamma$  in 0 to the left nor to the right then we can use the same argumentation for the first  $t \in [0, 1]$  in which  $\kappa$  leaves  $\gamma$ . The case  $\gamma(0) = \sigma(0)$  can be shown analogously.  $\square$

**Remark 14.**  $\Delta_q(\gamma, \kappa)$  is odd if and only if  $q \in \gamma$ , so a restriction of  $\gamma$  is on  $\gamma$ .

**Definition 15 (Alternating Sequence).** We call  $(a_1, a_2, \dots, a_n), n \in \mathbb{N}$ , an *alternating sequence of length  $n$*  of a connecting arc  $\gamma$  if  $a_1 \preceq_\gamma a_2 \prec_\gamma \dots \prec_\gamma a_n$ , if  $a_1, a_3, \dots$  are restrictions from the left and  $a_2, a_4, \dots$  are restrictions from the right or vice versa.

We call  $(a_1, a_2, \dots, a_n)$  *left-blocking* if  $a_1$  is a restriction from the left and *right-blocking*, otherwise.

Note that  $a_1$  is a starting restriction if  $a_1 = a_2$ .

**Lemma 16.** Let  $p \in P^\circ$ ,  $\gamma_1, \gamma_2 \in \Gamma(\sigma, p)$ ,  $\gamma_1 < \gamma_2$  and  $q \in (\gamma_1 \cup \gamma_2) \setminus \{p\}$ .

1. If  $\gamma_1(0) \prec_\sigma \gamma_2(0)$  and  $\gamma_1 \cap \gamma_2 \subset \{p\}$  then  $\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = 1$ .
2. If  $\gamma_1(0) = \gamma_2(0)$  then

$$\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = \begin{cases} 0, & \text{if } q = \gamma_1(0), \\ 1, & \text{otherwise.} \end{cases}$$

3. If  $\gamma_1(0) \prec_\sigma \gamma_2(0)$  and  $\gamma_1 \cap \gamma_2 \not\subset \{p\}$  then with  $\gamma_1 \cap_+ \gamma_2 =: \{t_1\}$  and  $\gamma_2 \cap_- \gamma_1 =: \{t_2\}$  we know:

$$\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = \begin{cases} 2, & \text{if } q = \gamma_1(t_1), \\ 1, & \text{if } q \in \gamma_1|_{[0, t_1)} \cup \gamma_2|_{[0, t_2)}, \\ 3, & \text{if } q \in \gamma_1|_{(t_1, 1)} \cup \gamma_2|_{(t_2, 1)}. \end{cases}$$

4. If  $\gamma_2(0) \prec_\sigma \gamma_1(0)$  then with  $\gamma_1 \cap_+ \gamma_2 =: \{t_1\}$  and  $\gamma_2 \cap_- \gamma_1 =: \{t_2\}$  we have

$$\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = \begin{cases} 0, & \text{if } q = \gamma_1(t_1), \\ -1, & \text{if } q \in \gamma_1|_{[0, t_1)} \cup \gamma_2|_{[0, t_2)}, \\ 1, & \text{if } q \in \gamma_1|_{(t_1, 1)} \cup \gamma_2|_{(t_2, 1)}. \end{cases}$$

PROOF. We just consider the case  $\gamma_2(0) \prec_\sigma \gamma_1(0)$  as the other cases can be proven analogously. To simplify notation, let  $S_1 := \gamma_1|_{[0, t_1)}$ ,  $S_2 := \gamma_2|_{[0, t_2)}$ ,  $S_3 := \gamma_1|_{(t_1, 1)}$ ,  $S_4 := \gamma_2|_{(t_2, 1)}$ ,  $S_5 := \{\gamma_1(t_1)\}$  and  $I := \{1, \dots, 5\}$ .

Let  $j, k \in I$  and let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be a simple arc spline such that  $\alpha(0) \in S_j$ ,

$\alpha(1) \in S_k$  and  $\alpha \cap S_l = \emptyset$  for  $l \in I \setminus \{j, k\}$  and such that the set  $\{t \in [0, 1] : \alpha(t) \in S_j\}$  is a single interval as well as  $\{t \in [0, 1] : \alpha(t) \in S_k\}$ . Then  $\Delta(\gamma, \alpha)$  can be easily computed. For example, suppose  $\alpha(0) \in S_1$  and  $\alpha(1) \in S_2$ .  $\alpha$  must leave  $\gamma_1$  either to the left or to the right. If  $\alpha$  leaves  $\gamma_1$  to the left, then  $\alpha$  has to approach  $\gamma_2$  from the right, so  $\Delta(\gamma_2, \alpha) - \Delta(\gamma_1, \alpha) = 1 - 1 = 0$ . If  $\alpha$  leaves  $\gamma_1$  to the right then  $\alpha$  has to approach  $\gamma_2$  from the left and we get  $\Delta(\gamma_2, \alpha) - \Delta(\gamma_1, \alpha) = (-1) - (-1) = 0$ . Hence, if  $\alpha(0) \in S_1, \alpha(1) \in S_2$  then the value of  $\Delta(\gamma_2, \alpha) - \Delta(\gamma_1, \alpha)$  is independent of the actual path. It can be easily checked that this is true for every  $j, k \in I$ . We show the values of  $\Delta(\gamma_2, \alpha) - \Delta(\gamma_1, \alpha)$  with  $\alpha(0) \in S_j, \alpha(1) \in S_k, j, k \in I$ , in the following denoted by  $\Delta_{j,k}$ , in Table 1.

	1	2	3	4	5
1	0	0	2	2	1
2	0	0	2	2	1
3	-2	-2	0	0	-1
4	-2	-2	0	0	-1
5	-1	-1	1	1	0

Table 1: Values of  $D_{j,k}$  with  $j$  the row and  $k$  the column

It is easy to verify that  $\Delta_{j,k} + \Delta_{k,l} = \Delta_{j,l}$  holds for any  $j, k, l \in I$ , which yields  $\Delta_{j_1, j_2} + \Delta_{j_2, j_3} + \dots + \Delta_{j_{n-1}, j_n} = \Delta_{j_1, j_n}$ ,  $n \in \mathbb{N}, j_1, \dots, j_n \in I$ . Now, let us consider  $\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa)$  with  $q \in S_k, k \in I$ . We know that  $\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = \Delta(\gamma_2, \kappa|_{[0, t_\kappa(q)]}) - \Delta(\gamma_1, \kappa|_{[0, t_\kappa(q)]})$ . Let  $t' \in [0, 1]$  be the first parameter such that  $\kappa(t') \in \gamma_1 \cup \gamma_2$ , and let  $\kappa(t') \in S_j, j \in I$ . As  $\kappa|_{[t', t_\kappa(q)]}$  can be split in parts satisfying the condition from above, we know that  $\Delta_q(\gamma_2, \kappa) - \Delta_q(\gamma_1, \kappa) = \Delta(\gamma_2, \kappa|_{[0, t']}) - \Delta(\gamma_1, \kappa|_{[0, t']}) + \Delta_{j,k}$ . Hence, the claim is easily verified for every case.  $\square$

**Lemma 17.** *Let  $p \in P^\circ$ ,  $\gamma_1, \gamma_2 \in \Gamma(\sigma, p)$  and let  $(a_1, a_2)$  be a left-blocking alternating sequence of  $\gamma_1$ . If  $\gamma_1 < \gamma_2$  then  $a_1$  is a violation from the left of  $\gamma_2$  or  $a_2$  is a violation from the right of  $\gamma_2$ .*

PROOF. If  $a_1 = a_2$  then we know that  $a_1$  is a starting restriction from the left and  $a_2 = \sigma(1)$  and this yields that  $\gamma_1$  is the maximal arc in  $\Gamma(\sigma, p)$ . As we assumed  $\gamma_1 < \gamma_2$ , we know that  $a_1 \neq a_2$ .

If  $\gamma_2(0) \prec_\sigma \gamma_1(0)$  then, with  $\gamma_1 \cap_+ \gamma_2 =: \{t\}$ , we know that  $t_1(a_1) < t$  or  $t < t_1(a_2)$ . By Lemma 16, we get  $\Delta_{a_1}(\gamma_2, \kappa) - \Delta_{a_1}(\gamma_1, \kappa) = -1$  if  $t_1(a_1) < t$  and  $\Delta_{a_2}(\gamma_2, \kappa) - \Delta_{a_2}(\gamma_1, \kappa) = 1$  if  $t < t_1(a_2)$ . This yields  $\Delta_{a_1}(\gamma_2, \kappa) = -2$  if  $t_1(a_1) < t$  and  $\Delta_{a_2}(\gamma_2, \kappa) = 2$ , otherwise.

If  $\gamma_1(0) \preceq_\sigma \gamma_2(0)$  then, by Lemma 16, we know that  $\Delta_{a_2}(\gamma_2, \kappa) - \Delta_{a_2}(\gamma_1, \kappa) > 0$ , as  $a_2 \in \gamma_1|_{(0,1)}$ . Hence,  $\Delta_{a_2}(\gamma_1, \kappa) = 1$  yields  $\Delta_{a_2}(\gamma_2, \kappa) > 1$ .  $\square$

Note that Lemma 17 holds analogously if  $(a_1, a_2)$  is a right-blocking alternating sequence and  $\gamma_2 < \gamma_1$ .

**Theorem 18.** *Let  $p \in P^\circ$ ,  $\gamma \in \Gamma(\sigma, p)$  and let  $(a_1, a_2, a_3)$  be an alternating sequence of  $\gamma$ . If  $\gamma \not\subset P$  then  $p$  is not visible.*

PROOF. Assume that there is a  $\gamma_2 \in \Gamma(\sigma, p)$  with  $\gamma_2 \subset P$ . As  $\gamma_1$  and  $\gamma_2$  are not equal and as “ $\leq$ ” is a total order we have either  $\gamma_1 < \gamma_2$  or  $\gamma_2 < \gamma_1$ . We know that  $(a_1, a_2)$  is a left-blocking alternating sequence and  $(a_2, a_3)$  is a right-blocking alternating sequence or vice versa. In either case we can apply Lemma 17 and get that there is a violation of  $\gamma_2$ .  $\square$

## 5. Algorithm

Recall that  $\kappa = \kappa_1 \sqcup \kappa_2 \sqcup \dots \sqcup \kappa_n$  is the channel boundary, an arc spline with  $n \in \mathbb{N}$  segment. Let  $p \in P^\circ$  and  $\gamma \in \Gamma(\sigma, p)$ . We say that a channel segment  $\kappa_j, j \in \{1, \dots, n\}$  is a *restriction from the left of  $\gamma$*  if  $\{-1\} \subseteq \{\Delta_q(\gamma, \kappa) : q \in \kappa_j\} \subseteq \{-1, 0\}$ . Analogously, we say that  $\kappa_j$  is a *restriction from the right of  $\gamma$*  if  $\{1\} \subseteq \{\Delta_q(\gamma, \kappa) : q \in \kappa_j\} \subseteq \{1, 0\}$ . We call  $\kappa_j$  a *violation from the left of  $\gamma$*  if there is a  $q \in \kappa_j$  with  $\Delta_q(\gamma, \kappa) < -1$  and a *violation from the right of  $\gamma$*  if there is a  $q \in \kappa_j$  with  $\Delta_q(\gamma, \kappa) > 1$ . With  $\kappa_j, \kappa_k, j, k \in \{1, \dots, n\}$  being two restrictions of  $\gamma$ , we write  $\kappa_j \prec_\gamma \kappa_k$  if there are Restrictions  $q_1 \in \kappa_j, q_2 \in \kappa_k$  with  $q_1 \prec_\gamma q_2$ . With this in mind, we can define an alternating sequence  $(\kappa_j, \kappa_k)$ . Note that a channel segment can not be both a restriction from the left and a restriction from the right at the same time.

In this chapter we give an algorithm and prove that it computes in linear time an arc  $\gamma \in \Gamma(\sigma, p)$  that is either a visibility arc or an arc having an alternating sequence of length three. To explain the rough idea of the algorithm, we skip the initialization for a moment: assume we have an arc  $\gamma \in \Gamma(\sigma, p)$  and indices  $L$  and  $R$  such that the channel segment  $\kappa_L$  is a restriction from the left of  $\gamma$  and we know that no segment  $\kappa_i, i \in \{1, \dots, L-1\}$  is a violation from the left and  $\kappa_R$  is a restriction from the right with the respective property. Then we check if  $\kappa_{L+1}$  is a violation from the left, if  $\kappa_{R+1}$  is a violation from the right, if  $\kappa_{L+2}$  is a violation from the left, and so forth. As soon as we find a violation, suppose a violation from the left, we update the index  $L$  and the arc  $\gamma$  such that  $\kappa_L$  is a restriction from the left of  $\gamma$  and  $\kappa_R$  remains a restriction from the right. Then we repeat the procedure starting at the updated indices  $L$  and  $R$ . Initially,  $\gamma$  is the minimal arc in  $\Gamma(\sigma, p)$  and  $L = R = 0$ . We will see that this algorithm is correct as there do not appear new “relevant” restrictions before  $\kappa_L$  or  $\kappa_R$ , respectively. Since we check alternately for violations from the left and from the right and since every step can be computed in constant time the overall runtime is linear.

**Theorem 19.** *Algorithm 1 is correct, this means that the result is a visibility arc or an arc in  $\Gamma(\sigma, p)$  having an alternating sequence of length three.*

The proof is divided into two parts: first we show that the arcs in line 16 and 19 exist. This will be done in Lemma 20. Afterwards, we prove an invariant of the algorithm, see Lemma 21, which yields that the arc in line 23 has no violations and thus is a visibility arc, see Lemma 22.

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**Algorithm 1**

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1: Input: a channel with  $n$  segments and a point  $p$  inside the channel
2: Output:  $\gamma \in \Gamma(\sigma, p)$  that is a vis. arc or has an alt. sequ. of length three
3: Let  $\gamma = \min \Gamma(\sigma, p)$ 
4: // segment number of current restr. from the left,  $L$ , or right,  $R$ ;  $\kappa_0 := \sigma$ 
5:  $L = R = 0$ 
6:  $l = r = 0$  // current indices of segments to check for violations
7: while  $l < n$  or  $r < n$  do
8:   Let  $l = \min(l + 1, n)$ ;  $r = \min(r + 1, n)$ 
9:   if  $\gamma$  has an alternating sequence of length three with restrictions in
       $\sigma, \kappa_L, \kappa_{L+1}, \dots, \kappa_l, \kappa_R, \kappa_{R+1}, \dots, \kappa_r$  then
10:     return  $\gamma$ 
11:   else if there is a  $\gamma^* \in \Gamma(\sigma, p)$  with alternating sequence of length three
      in  $\sigma, \kappa_L, \kappa_R, \kappa_l, \kappa_r$  then
12:     return  $\gamma^*$ 
13:   end if
14:   // Update  $\gamma$  in case of a violation
15:   if  $\kappa_l$  is a violation from the left of  $\gamma$  then
16:     Choose  $\gamma \in \Gamma(\sigma, p)$  as the arc with right-bl. alt. sequence  $(\kappa_R, \kappa_l)$ 
17:      $L = l$ ;  $r = R$ 
18:   else if  $\kappa_r$  is a violation from the right of  $\gamma$  then
19:     Choose  $\gamma \in \Gamma(\sigma, p)$  as the arc with right-bl. alt. sequence  $(\kappa_r, \kappa_L)$ 
20:      $R = r$ ;  $l = L$ 
21:   end if
22: end while
23: return  $\gamma$ 
```

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**Lemma 20.** *The arcs in line 16 and 19 of Algorithm 1 exist and are unique.*

PROOF. We assume that for any three distinct points  $p_1, p_2, p_3$  there exists an arc with starting point  $p_1$ , endpoint  $p_3$  that passes  $p_2$ . This can be done since the maximal length of relevant arcs can be limited, cf. Remark 9.

Furthermore, we assume that  $p \in P^\circ$  is strictly left of  $\sigma$ , which implies that the only intersection of a  $\gamma \in \Gamma(\sigma, p)$  and  $\sigma$  is  $\gamma(0)$ . If  $p$  is not strictly left of  $p$  then the proof works analogously, but it has to be considered that  $\gamma$  and  $\sigma$  might have two intersections. Furthermore, we only consider line 16 and we assume  $R \neq 0$  as the other cases can be proven similarly.

First, we prove the existence of an arc with right blocking alternating sequence  $(\kappa_R, \kappa_l)$ . Let  $\gamma_1$  denote the arc  $\gamma$  computed so far in the algorithm, let  $t_1 := t_\sigma(\gamma_1(0))$  and consider the function  $f : [0, t_1] \rightarrow \Gamma$  where  $f(t) = \gamma_t$  is the unique arc with starting point  $\sigma(t)$ , endpoint  $p$  such that  $\kappa_R \cap \gamma_t \neq \emptyset$  and  $\kappa_R$  is locally right of  $\gamma_t$  at every  $x \in \kappa_R \cap \gamma_t$ . Let  $t_2 := \min\{t \in [0, t_1] : f(t') \in \Gamma(\sigma, p) \text{ for all } t' \in [t, t_1]\}$  and  $\gamma_2 := f(t_2)$ . As  $f$  is continuous and  $f(t_1) = \gamma_1 \in \Gamma(\sigma, p)$ , we know that  $t_2$  is well defined. Since we assumed that  $p$  is left of  $\sigma$ , the only way to lose the connecting arc property is that the arc does not leave  $\sigma$  in 0 to the left. By continuity of  $\sigma' : [0, 1] \rightarrow S^1$  and  $(f(t))'(0) : [0, 1] \rightarrow S^1$ , we know that  $t_2 = 0$ ,  $\sigma'(t_2) = -\gamma_2'(0)$  or  $\sigma'(t_2) = \gamma_2'(0)$ . Since  $\sigma'(t_2) = -\gamma_2'(0)$  would yield at least three intersections of  $\gamma_1$  and  $[\gamma_2]$ , we know that  $t_2 = 0$  or  $\sigma'(t_2) = \gamma_2'(0)$ . So,  $\gamma_2$  has a starting restriction from the left.

Now we show that  $\kappa_R$  is a restriction from the right of every  $\gamma_t, t \in [t_2, t_1]$  and that  $\gamma_1 < \gamma_t$ : since  $\gamma_1 \cap \kappa_R \neq \emptyset$  and  $\gamma_t \cap \kappa_R \neq \emptyset$  and  $\kappa_R$  is locally right of both  $\gamma_1$  and  $\gamma_t$  at every intersection point it is easy to see that  $\gamma_t \cap_- \gamma_1 \neq \emptyset$ , which yields  $\gamma_1 < \gamma_t$ . Furthermore, with  $\gamma_t \cap_- \gamma_1 := \{\tau_t\}$  and  $\gamma_1 \cap_+ \gamma_t := \{\tau_1\}$ , i.e.  $\gamma_1(\tau_1) = \gamma_t(\tau_t)$ , we know that  $\kappa_R \cap \gamma_t|_{[0, \tau_t]} = \kappa_R \cap \gamma_1|_{(\tau_1, 1]} = \emptyset$ . Let  $q \in \kappa_R$  be a restriction from the right of  $\gamma_1$ . By Lemma 16, we get  $\Delta_q(\gamma_t, \kappa) = 1$  if  $q = \gamma_1(\tau_1)$  and  $\Delta_q(\gamma_t, \kappa) = 0$  if  $q = \gamma_1|_{[0, \tau_1]}$ . Hence, as  $\kappa_R$  is locally right of  $\gamma_t$  at every point in  $\kappa_R \cap \gamma_t$ , in either case we know that for every  $q \in \kappa_R$  we have  $\Delta_q(\gamma_t, \kappa) \in \{0, 1\}$  with value 1 if and only if  $q \in \gamma_t$ . This yields that  $\kappa_R$  is a restriction from the right of  $\gamma_t$ . Analogously, one can show that every  $q \in \kappa_l$  with  $\Delta_q(\gamma_1, \kappa) = -2$  is a restriction from the left of  $\gamma_t, t \in [t_2, t_1]$  if  $q \in \gamma_t$ .

We know that there are  $q_1 \in \kappa_R, q_2 \in \kappa_l$  such that  $(q_1, q_2)$  is a right-blocking alternating sequence of  $\gamma_1$  as otherwise there would be an alternating sequence of length three in  $\kappa_L, \kappa_R, \kappa_l$ . Since  $f$  is continuous and  $\kappa_l$  and  $\kappa_R$  are closed, we know that there is a minimal  $t^* \in [t_2, t_1]$  such that  $\gamma_4 := f(t^*)$  has a right-blocking alternating sequence  $(q_1, q_2)$  with  $q_1 \in \kappa_R, q_2 \in \kappa_l$ . If  $t^* = t_2$  then  $\gamma_4$  has an alternating sequence of length three since it has a starting restriction from the left. Otherwise, we know that the above property can only be lost if the touching point on  $\kappa_R$  “jumps” such that it is after the one on  $\kappa_l$  with respect to the respective arc or the intersection with  $\kappa_l$  that is after  $\kappa_R$  with respect to the respective arc disappears. In the first case, we know that there is a  $q \in \kappa_l$  such that  $(\kappa_R(0), q, \kappa_R(1))$  or  $(\kappa_R(1), q, \kappa_R(0))$  is a right-blocking alternating sequence of  $\gamma_4$ . Otherwise, we know that  $\kappa_l$  is locally left of  $\gamma_4$  at

$q_1$ . So,  $\kappa_l$  is a restriction from the left of  $\gamma_4$  or there are  $q_1 \in \kappa_l$ ,  $q_2 \in \kappa_R$ , such that  $(q_1, q_2, \kappa_l(0))$  or  $(q_1, q_2, \kappa_l(1))$  is a left-blocking alternating sequence of  $\gamma_4$ . If  $\kappa_l$  is a restriction from the left of  $\gamma_4$ , we found an arc with right-blocking alternating sequence  $(\kappa_R, \kappa_l)$ .

To show uniqueness we assume that there are two arcs  $\gamma_1, \gamma_2 \in \Gamma(\sigma, p)$ ,  $\gamma_1 \neq \gamma_2$  with right-blocking alternating sequence  $(\kappa_R, \kappa_l)$ . Since  $\kappa_R$  is a restriction from the right of  $\gamma_1$  and  $\gamma_2$ , we know that  $\kappa_R \cap \gamma_1 \neq \emptyset$  and  $\kappa_R \cap \gamma_2 \neq \emptyset$  and that  $\kappa_R$  is locally right of both  $\gamma_1$  and  $\gamma_2$  at every point of the respective intersection. Hence, we know that  $\kappa_R \subset \bar{Z}$  with  $Z$  a connected component of  $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \sigma)$ . Similarly, this holds for  $\kappa_l$ . Distinguishing by the order of  $\gamma_1$  and  $\gamma_2$  and by the order of  $\gamma_1(0)$  and  $\gamma_2(0)$  with respect to  $\prec_\sigma$  it is easy to see that this is not possible.  $\square$

**Lemma 21.** *Let  $q_R$  be the first point on  $\kappa_R$ , with respect to  $\kappa$ , such that  $q_R \in \gamma$  and let  $q_L$  be the first point on  $\kappa_L$ , with respect to  $\kappa$ , such that  $q_L \in \gamma$ . Then we have:*

1. *if there is a restriction from the right  $q$  with  $q \prec_\kappa q_R$  and  $q \prec_\gamma q_R$  then, with  $q$  the first such restriction with respect to  $\kappa$ , we know that  $\kappa$  approaches  $\gamma$  in  $t_\kappa(q)$  from the left.*
2. *if there is a restriction from the left  $q$  with  $q \prec_\kappa q_L$  and  $q_L \prec_\gamma q$  then, with  $q$  the first such restriction with respect to  $\kappa$ , we know that  $\kappa$  approaches  $\gamma$  in  $t_\kappa(q)$  from the right.*

PROOF. We only prove the first invariant, the second one can be shown analogously. Obviously, the invariant holds initially. Consider the update step in line 19 and 20 and suppose  $L \neq 0$  and  $R \neq 0$ . The update in line 16 and 17 and the cases with vanishing  $L$  or  $R$  can be proven analogously. Let  $\gamma_1$  and  $R_1$  be the arc  $\gamma$  and the value of  $R$  computed so far in the algorithm and denote the respective values after the update by  $\gamma_2$  and  $R_2$ . Furthermore, let  $r_1$  be the first point on  $\kappa_{R_1}$ , with respect to  $\kappa$ , such that  $r_1 \in \gamma_1$ , let  $r_2$  be the first point on  $\kappa_{R_2}$  such that  $r_2 \in \gamma_2$  and let  $r_3$  be the first point on  $\kappa_{R_2}$  such that  $r_3 \in \gamma_1$ . Assume that the invariant holds before the update and that it is violated after the update. Then we know that there is a  $q \in \kappa$  that is a restriction from the right of  $\gamma_2$  and that satisfies  $q \prec_\kappa r_2$  and  $q \prec_\gamma r_2$ . With  $q$  the first such restriction with respect to  $\kappa$ , we know that  $\kappa$  approaches  $\gamma_2$  in  $t_\kappa(q)$  from the right. Since  $\gamma_1 < \gamma_2$  and  $q \prec_{\gamma_2} r_2 \prec_{\gamma_2} \gamma_2(t)$ , with  $\{t\} := \gamma_2 \cap_- \gamma_1$ , Lemma 16 yields  $\Delta_q(\gamma_1, \kappa) = 2$ . Hence,  $q \prec_\kappa r_1$  as otherwise  $\gamma$  and  $R$  would have been updated earlier.

Consider the closed path

$$\alpha := \gamma_1|_{[0, t(r_3)]} \sqcup \kappa|_{[t(r_3), t(r_2)]} \sqcup \overline{\gamma_2|_{[0, t(r_2)]}} \sqcup \sigma|_{[t(\gamma_2(0)), t(\gamma_1(0))]}$$

and let  $Z$  be the connected component of  $\mathbb{R}^2 \setminus \alpha$  that is locally right of  $\gamma_2$  in  $q$ . Obviously,  $\kappa(0) \notin Z$  and as  $\kappa$  approaches  $\gamma_2$  in  $t_\kappa(q)$  from the right, we know that there is an  $\varepsilon > 0$  such that  $\kappa([t_\kappa(q) - \varepsilon, t_\kappa(q))) \subset Z$ . Hence, there must be a  $t \in [0, t_\kappa(q))$  such that  $\kappa(t) \in \alpha$ .



First, we consider the case  $r_3 \prec_{\gamma_1} r_1$ . We know that there is a  $t \in [0, t_\kappa(q)]$  with  $\kappa(t) \in \gamma_1|_{[0, t(r_3)]} \cup \gamma_2|_{[0, t(r_2)]}$ . Let  $t^* \in [0, t_\kappa(q)]$  be maximal such that  $\kappa(t^*) \in \gamma_1|_{[0, t(r_3)]} \cup \gamma_2|_{[0, t(r_2)]}$  and let  $q^* = \kappa(t^*)$ . Then,  $\kappa|_{(t^*, t_\kappa(q))} \subset Z$  and in particular  $\kappa|_{(t^*, t_\kappa(q))} \cap \gamma_2 = \emptyset$ . Hence,  $q^* \notin \gamma_2|_{[0, t(r_2)]}$ , since otherwise  $q^*$  would be a restriction from the right, which would be a contradiction as  $q$  was defined as the first such restriction. So,  $q^* \in \gamma_1|_{[0, t(r_3)]}$ . Since  $\kappa|_{(t^*, t_\kappa(q))} \cap \gamma_2 = \emptyset$  and  $\kappa$  approaches  $\gamma_2$  in  $t_\kappa(q)$  from the right, we know that  $\Delta_{q^*}(\gamma_2, \kappa) = 0$ . Hence, Lemma 16 yields  $\Delta_{q^*}(\gamma_1, \kappa) = 1$ , so  $q^*$  is a restriction from the right of  $\gamma_1$ . As  $q^* \prec_\kappa r_1$  and  $q^* \prec_{\gamma_1} r_3 \prec_{\gamma_1} r_1$ , we can define  $q'$  to be the first restriction from the right of  $\gamma_1$  with respect to  $\kappa$  such that  $q' \prec_{\gamma_1} r_1$  and we know  $q' \prec_\kappa r_1$ . Since the invariant with respect to  $\gamma_1$  holds, we know that  $\kappa$  approaches  $\gamma_1$  in  $t_\kappa(q')$  from the left. With the same arguments as before we can conclude that there must be a  $t \in [0, t_\kappa(q')]$  with  $\kappa(t) \in \gamma_1|_{[0, t(r_3)]} \cup \gamma_2|_{[0, t(r_2)]}$  and the point corresponding to the last such parameter is either a restriction from the right of  $\gamma_1$  or a restriction from the right of  $\gamma_2$ . In either case this yields a contradiction. In the case  $r_1 \prec_{\gamma_1} r_3$  the argumentation is basically the same. Since  $\kappa|_{[0, t_\kappa(q)]} \cap \kappa|_{[t(r_1), t(r_3)]} = \emptyset$  and  $\kappa|_{[t(r_1), t(r_3)]}$  does not cut  $\gamma_1|_{[t(r_1), t(r_3)]}$ , we also know that there is a  $t \in [0, t_\kappa(q)]$  with  $\kappa(t) \in \gamma_1|_{[0, t(r_1)]} \cup \gamma_2|_{[0, t(r_2)]}$ . Let  $t^* \in [0, t_\kappa(q)]$  be maximal such that  $\kappa(t^*) \in \gamma_1|_{[0, t(r_1)]} \cup \gamma_2|_{[0, t(r_2)]}$  and let  $q^* = \kappa(t^*)$ . The only difference to the case  $r_3 \prec_{\gamma_1} r_1$  is that it not obvious that  $q^*$  is a restriction from the right of  $\gamma_1$  or  $\gamma_2$  if  $q^* \in \gamma_1|_{[0, t(r_1)]}$  or  $\gamma_2|_{[0, t(r_2)]}$ , respectively. We only show that  $q^*$  is a restriction from the right of  $\gamma_2$  if  $q^* \in \gamma_2|_{[0, t(r_2)]}$  as the other cases can be proven analogously. We know that there is no restriction from the left in  $\kappa|_{[t_\kappa(r_1), t_\kappa(r_3)]} \cap \gamma_1|_{[t_1(r_1), t_1(r_3)]}$  and we assume  $\kappa|_{(t_\kappa(r_1), t_\kappa(r_3))} \cap \gamma_1|_{(t_{\gamma_1}(r_1), t_{\gamma_1}(r_3))} = \emptyset$ , otherwise  $\kappa|_{[t_\kappa(r_1), t_\kappa(r_3)]}$  can be split such that there is no intersection at each part and the arguments work for every part. Let

$$\alpha_2 := \kappa|_{[t_\kappa(r_1), t_\kappa(r_3)]} \sqcup \overline{\gamma_1|_{[t_{\gamma_1}(r_1), t_{\gamma_1}(r_3)]}}$$

and let  $Z_2$  be the connected component locally right of  $\gamma_1|_{[t_{\gamma_1}(r_1), t_{\gamma_1}(r_3)]}$ . Then we know that  $\kappa|_{(t^*, t_\kappa(q))} \subset (Z \cup Z_2 \cup \gamma_1|_{[t_{\gamma_1}(r_1), t_{\gamma_1}(r_3)]})$ . Since  $\Delta(\gamma_2, \beta) = 0$  for every arc spline  $\beta \subset Z$  and since we can split  $\kappa|_{[t^*, t_\kappa(q)]}$  such that every part is either in  $Z \cup \gamma_1|_{[t_{\gamma_1}(r_1), t_{\gamma_1}(r_3)]}$  or in  $Z_2 \cup \gamma_1|_{[t_{\gamma_1}(r_1), t_{\gamma_1}(r_3)]}$  it is enough to show that  $\Delta(\gamma_2, \beta) = 0$  for any arc spline  $\beta$  with  $\beta(0), \beta(1) \in \gamma_1|_{(t_{\gamma_1}(r_1), t_{\gamma_1}(r_3))}$  and  $\beta^\circ \in Z_2$ . We assume that  $Z_2$  is the interior of  $\alpha_2$ , otherwise it works analogously. As  $\alpha_2$  leaves  $\gamma_1$  in 0 to the right, we know that there is a  $\varepsilon > 0$  such that  $\gamma_1|_{[t_{\gamma_1}(r_1) - \varepsilon, t_{\gamma_1}(r_1)]} \cap Z_2 = \emptyset$ . For every  $\tilde{q} \in \kappa|_{[t_\kappa(r_1), t_\kappa(r_3)]}$  we know that  $\tilde{q}$  is not a violation from the right of  $\gamma_1$  and if  $\tilde{q} \in \gamma_1|_{[0, t_{\gamma_1}(r_1)]}$  then it is not a restriction from the left of  $\gamma_1$ . This yields  $\Delta(\gamma_1|_{[0, t_{\gamma_1}(r_1) - \varepsilon]}, \alpha_2) = 0$ . Hence, with Proposition 12, we know that  $\gamma_1(0) \notin Z_2$  and  $\gamma_1(1) \notin Z_2$  since  $\Delta(\gamma_1, \alpha_2) = 0$ . Now, let  $\beta$  be an arc spline with  $\beta(0), \beta(1) \in \gamma_1|_{(t_{\gamma_1}(r_1), t_{\gamma_1}(r_3))}$

and  $\beta^\circ \in Z_2$ . Then, the interior of  $\beta$ , denoted by  $I_\beta$ , is a subset of  $Z_2$ . Hence,  $\gamma_1(0), \gamma_1(1) \notin I_\beta$  and with Proposition 12 we get  $\Delta(\gamma_1, \beta) = 0$ . Lemma 16 yields  $\Delta(\gamma_2, \beta) = 0$  as  $\beta$  is supposed to be a part of the channel boundary.  $\square$

**Lemma 22.** *The arc  $\gamma$  in line 23 is a visibility arc.*

PROOF. Let  $t^* \in [0, 1]$  maximal such that there is no violation in  $\kappa|_{[0, t^*]}$ . Suppose  $t^* < 1$ . Then we know that  $\kappa(t^*)$  is a restriction of  $\gamma$ . We suppose that  $\kappa(t^*)$  is a restriction from the right of  $\gamma$  as it works analogously, otherwise. Let  $q$  be the first restriction from the right of  $\gamma$ , with respect to  $\kappa$ . We know that  $\kappa$  approaches  $\gamma$  in  $t_\gamma(q)$  from the right. With  $q_r$  the first restriction from the right in  $\kappa_R$ , we know that  $q \prec_\kappa \kappa(t^*) \prec_\kappa q_r$  which yields  $q_r \prec_\gamma q$  because of Lemma 21. Consider the closed arc spline  $\alpha := \kappa|_{[0, t_\kappa(q)]} \sqcup \gamma|_{[0, t_\gamma(q)]} \sqcup \sigma|_{[t_\sigma(\gamma(0)), 1]}$  and the connected component  $Z$  of  $\mathbb{R}^2 \setminus \alpha$  that is locally right of  $\gamma|_{[0, t_\gamma(q)]}$ . We know that there is no  $t \in [t_\kappa(q_r), 1]$  such that  $\kappa$  leaves  $\gamma|_{[0, t_\gamma(q)]}$  in  $t$  to the left as otherwise there would be a violation from the right. This yields  $\kappa|_{[t_\kappa(q_r), 1]} \subset \bar{Z}$  as  $\kappa$  is simple and  $\kappa^\circ \cap \sigma = \emptyset$ . If  $\gamma(0) \neq \sigma(0)$  then obviously  $\kappa(1) = \sigma(0) \notin \bar{Z}$  and we get a contradiction. Otherwise, if  $\gamma(0) = \sigma(0)$  then  $\Delta(\gamma, \kappa) = 1$ , a contradiction to Lemma 13.  $\square$

**Theorem 23.** *Algorithm 1 can be implemented such that its runtime is linear with respect to  $n$ , the number of segments of the channel boundary.*

PROOF. To reach linear runtime we have to keep a record of  $\Delta_{\kappa_l(0)}(\gamma, \kappa)$  and  $\Delta_{\kappa_r(0)}(\gamma, \kappa)$ . Initially the value is clear. If  $\gamma$  is not updated in an iteration then  $\Delta_{\kappa_l(0)}(\gamma, \kappa)$  can easily be updated as  $\Delta_{\kappa_l(0)}(\gamma, \kappa) = \Delta_{\kappa_{l-1}(0)}(\gamma, \kappa) + \Delta(\gamma, \kappa_{l-1})$ . If  $\gamma$  is updated then the value of  $\Delta_{\kappa_l(0)}(\gamma, \kappa)$  is known as  $\kappa_l$  is a restriction from the left. All this operations can be computed in constant time. The same holds for  $\Delta_{\kappa_r(0)}(\gamma, \kappa)$ , respectively.

As  $\kappa_L$  and  $\kappa_R$  are restrictions, with Lemma 16 we can compute  $\Delta_q(\gamma^*, \kappa)$  with  $\gamma^* \in \Gamma(\sigma, p)$ ,  $q \in \kappa_L, \kappa_R, \kappa_l, \kappa_r$  only considering the respective segment. With  $\lambda$  a channel segment,  $\Delta_{\lambda(t)}(\gamma, \kappa)$  as a function of  $t$  is locally constant for every  $t \in [0, 1]$  with  $\lambda(t) \notin \gamma$ . This yields that the value changes at no more than two intersection points, hence  $\{\Delta_q(\gamma, \kappa) : q \in \lambda\}$ ,  $\lambda = \kappa_l, \kappa_r, \kappa_L, \kappa_R$  can be computed in constant time which yields that the statements in line 15 and 18 can be computed in constant time. The check in line 11 and the arcs in line 16 and 19 can also be computed in constant time using the problem of Apollonius, cf. [4]. To compute line 9 in constant time, it is enough to keep a record of the restriction from the left of  $\gamma$  in  $\kappa_L, \kappa_{L+1}, \dots, \kappa_l, \kappa_R, \kappa_{R+1}, \dots, \kappa_r$  that is minimal with respect to  $\gamma$  and the maximal restriction from the right, respectively. The minimal or maximal restriction can be updated in constant time as only the current segment  $\kappa_l$  or  $\kappa_r$  must be taken into account. It is obvious that the remaining statements can be computed in constant time.

To get a linear runtime we show that the loop starting in line 7 is passed at most  $2n$  times. Therefore, we show that in every iteration at least one segment

is finally processed. In each iteration in that the algorithm has not stopped, we can distinguish two cases: neither  $\kappa_l$  nor  $\kappa_r$  is a violation from the left or right or at least one of them,  $\kappa_l$  or  $\kappa_r$ , is a violation from the left or right. In the first case  $l$  and  $r$  are incremented by one. Otherwise, if  $\kappa_l$  is a violation from the left,  $r$  is reset to  $R$  but  $L$  is incremented to  $l$ , so the segments  $\kappa_{L+1}, \dots, \kappa_l$  are finally processed. Analogously, if  $r - R = k$  and  $\kappa_r$  is a violation from the right then this  $k$  segments are finally processed. As the algorithm stops if both  $l$  and  $r$  are equal to  $n$  we know that there are at most  $2n$  iterations.  $\square$

**Remark 24.** The result of Algorithm 1 is either a visibility arc or an arc that has an alternating sequence of length three and thus proves that the considered point is not visible. Even if we consider numerical errors, in either case we get an arc which proves that the respective point is visible or not visible up to a certain tolerance. Hence, numerical inaccuracies only affect situations where some uncertainty is inevitable.

## 6. Conclusion

We treat the problem if a point inside a simple closed arc spline is circularly visible from a boundary arc. In particular, we provide an easy to check criterion that implies that the point is not visible and we present a simple and numerically stable linear time algorithm that checks visibility.

We will integrate the results into the SMAP approach, see [6]. Thus, point sequences can be approximated by an arc spline up to a arbitrary tolerance with optimal segment number in a numerically stable and efficient manner.

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